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Series Expansion for Mesonic Masses In Multicolor QCD

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ABSTRACT

A systematic procedure for the calculation of mesonic masses in QCD with large number of colors is proposed. The masses are expanded in terms of an auxiliary parameter α, which is set to 1 at the end of the calculation. The expansion coefficients are expressed in general in terms of the Feynman integrals of QCD.

The terms of the order α^n involve diagrams with n loops. Explicit expressions are found up to α^3. These terms appear to be rather small, so that one may try to extrapolate to $\alpha=1$. The qualitative properties of the spectrum are plausible, but for quantitative predictions further calculations are required.

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INTRODUCTION

The hadronic spectrum in massless QCD is essentially universal. Due to renormalizability, there is only one scale

$$\mu = \Lambda \exp\left(-\frac{1}{a g_0^2} - b \ln(a g_0^2)\right) \tag{1}$$

where $\Lambda \to \infty$ is the U.V. cutoff, $g_0 \to 0$ is the corresponding bare coupling constant, and a and b are the well-known coefficients of the β-function.

For finite g_0 there are also terms $O(g_0^2)$ in the exponential of (1), coming from 3 and more loop terms in β-function, but in the limit $g_0 \to 0$ those terms drop out. Note, however, that μ is defined up to a common scale factor, since by changing $g_0^2 \to g_0^2 + z_1 g_0^4 + z_2 g_0^6 + \ldots$ (which corresponds to the change of renormalization procedure), we achieve the factor in (1), and this factor remains finite in the limit $g_0 \to 0$.

The effective coupling at the scale μ is a universal number and cannot be treated as a free parameter. (In folklore that μ is referred to as a position of a logarithmic pole of the effective coupling but the coupling may always be redefined as to avoid the fictitious pole.)

As for the quark masses, for the light quarks u, d, s, the masses arise almost spontaneously - the bare masses are likely to be of electromagnetic origin. If we disregard the charmed and ψ particles, and neglect the bare masses of u, d, s, quarks, then we are left with μ

as the single parameter of the theory. The spectrum would then be

$$M_i = \mu \times \text{(universal numbers)} \ . \tag{2}$$

These universal numbers depend only on discrete parameters of the theory -- the number $N_c = 3$ of colors and number $N_f = 3$ of flavors. As was first noticed by G. 't Hooft,[1] the theory simplified in the multicolor limit

$$N_c \to \infty \ . \tag{3}$$

In this limit all dimensionless quantities can be expanded in $1/N_c$, and in particular

$$\frac{M_i}{\mu} = A_i + B_i/N_c + \ldots \tag{4}$$

As for the general scale μ, it will remain finite in the multicolor limit provided the product $N_c g_o^2$ is kept fixed. The coefficients N_c^{-1} a and b tend to the following limits

$$\frac{a}{N_c} \to \frac{11}{48\pi^2} \tag{5}$$

$$b \to \frac{51}{121} \ . \tag{6}$$

Estimates,[2] based on the unitarity condition, show that all the mesons are stable at infinite number of colors, so that the coefficients A_i in (4) are real. The amplitudes of decay of meson to other mesons are down by some powers of N_c. This is due to the fact that colorless states represent a small fraction of all kinematically allowed states:

N_c^{-1} for the $\bar{q}q$ states, N_c^{-2} for $\bar{q}\bar{q}qq$ states etc. The decays into quarks and gluons are favorable kinematically, but we expect that these decays are forbidden dynamically.

We are not going to discuss now the arguments in favor of color confinement, but rather accept confinement as a working hypothesis. Within the framework, proposed below, this hypothesis looks quite natural and to a certain extent appears to be self consistent.

II. INFRARED REGULARIZATION AND α-EXPANSION

How shall we calculate the universal coefficients A_i, B_i, in the mesonic masses? It seems unlikely, that they can be exactly expressed on terms of known mathematical functions. So, special numerical methods are to be developed. Here we propose a method which resembles the ϵ-expansion in the theory of phase transitions. An auxiliary parameter α is introduced in such a way, that at small α some kind of perturbation theory can be applied, whereas at $\alpha=1$ one returns to original QCD.

The coefficients A_i, B_i, etc. of expansion of mesons masses in $1/N_c$ will now depend on α and can in turn be expanded in a power series. There is some evidence (see below), that α-expansion is convergent, in contrast with $1/N_c$ expansion. The coefficients of the α-expansion involve the Feynman integrals of ordinary perturbation theory (up to n loops for α^n). The explicit relations will be given up to α^3 but there is no real problem in calculation of higher coefficients. As for the scale μ, it remains unchanged, so that the α-expansion does not alter the famous nonanalytic relation (1) between the scale of masses and the coupling constant.

We start from the Pade regularization of multicolor QCD,[3,4] where the masses M_i depend on the infrared cutoff $R \to \infty$ in the following way

$$M_i = \frac{1}{R} F_i(\mu R) \tag{7}$$

The $(\text{masses})^2$ are defined as the positions of poles of 2-point functions of local gauge invariant fields with corresponding quantum numbers (currents, etc.). The matrix N/N Pade approximation in k^2 near $k^2 = -\Lambda^2$ is applied, and R is defined as N/Λ. At $R \to \infty$ we should recover the original theory, whereas at finite R we have a discrete factorizable spectrum, free of ghosts and tachyons (see Refs. 3 and 4 for more details).

An important property of Pade regularization is that it always overestimates the masses. They decrease monotonously with N(i.e., with R). This follows from general properties of Pade approximant of Stiljes functions (the functions with positive spectral measure in $0<k^2<\infty$ in our case.) Our 2-point functions are Stiltjes functions due to the spectral conditions.

The functions F_i in (7) are known as a power series in the effective coupling ag_R^2, which tends to zero at $\mu R << 1$ as

$$ag_R^2 \to (-\xi + b\,\ell n(-\xi))^{-1} \tag{8}$$

$$\xi = \ell n(\mu R) \ . \tag{9}$$

We are interested in the opposite limit, where the effective coupling is not small, and we expect that $F_i(x)$ increases linearly with x, so that the masses and mass spacings remain finite at infinite cutoff R. This would correspond to color confinement. The other alternative is that the masses condense to the thresholds of $\bar{q}q$ and/or gluon production, so that the spectrum becomes continuous.

Without asymptotic freedom (say, in massless QED), this condensation would certainly occur, since then the coupling decreases at $R \to \infty$, so that F_i tends to the zeroth order terms, which are finite.

Also, if we take a finite number of colors in QCD, the masses would condense to the physical thresholds (which are at zero mass due to the Goldstone theorem). Thus, only in QCD with infinite number of colors the above definition of masses would yield a finite limit at $R \to \infty$. (Nevertheless, it is possible to generalize the method as to incorporate $1/N_c$ corrections. This will be considered in a subsequent paper.)

Let us now generalize the theory by introducing a continuous parameter α as follows

$$M_i(\alpha, R) = \frac{\mu}{(\mu R)^{\alpha^2}} F_i(\mu R) \tag{10}$$

If the function $F_i(x)$ increases linearly, then at $\alpha^2 < 1$ the generalized mass would increase at $\mu R \to \infty$. On the other hand, at $\mu R \to 0$ F_i tends to a constant, so that the mass increases again. Thus there should be a minimum at some μR depending on α. When α tends to 1, this minimum shifts to $\mu R = \infty$, as one can see from Fig. 1, where $\ell n(M_i/\mu)$ is plotted versus $\ell n\, \mu R$ for various α and from Fig. 2, where $\ell n\, \mu R$ is plotted versus α.

At $\alpha \to 0$ the minimum corresponds to small μR, where perturbation theory can be applied. This happens since at small α the factor $(\mu R)^{-\alpha^2}$ in (10) decreases very slowly with R, so that the small increase of $F_i(\mu R)$ due to the increase of the effective coupling g_R would compensate the above mentioned decrease. As we shall see below, the corresponding value of g_R^2 is proportional to α, and hence the perturbation theory would yield an α-expansion.

Let us proceed in a systematic way. It is convenient to take the log of (10)

$$\ell n \frac{M_i(\alpha, R)}{\mu} = f_i(\xi) - \alpha^2 \xi \ . \tag{11}$$

Here

$$f_i(\xi) \equiv \ell n\, F_i(e^{\xi}) \ . \tag{12}$$

The minimum of the r.h.s. of (11) is nothing but a Legendre transform of $f(\xi)$:

$$\phi_i(\alpha) = \min[\, f_i(\xi) - \alpha^2 \xi \,] \tag{13}$$

$$\frac{d\phi}{d\alpha} = -2\alpha\xi \tag{14}$$

$$\frac{df_i}{d\xi} = \alpha^2 \tag{15}$$

In order to find the masses, we should calculate this Legendre transform at $\alpha=1$

$$\ell n \frac{M_i}{\mu} = \phi_i(1) \quad . \tag{16}$$

Suppose, we know the expansion of masses in effective coupling g_R, i.e., we know $f_i(\xi)$ in the form

$$f_i = a_i + b_i ag_R^2 + c_i(ag_R^2)^2 + d_i(ag_R^2)^3 + \ldots \tag{17}$$

The relation between g_R and $\xi = \ell n\mu R$ is given by the Gell-Mann-Low equation

$$\xi = \ell n \frac{\mu}{\Lambda} - \int_{g_o^2}^{g_R^2} dg^2/\beta(g^2) = -\frac{1}{ag_R^2} - b\ \ell n\ ag_R^2 - \psi(ag_R^2) + \psi(ag_o^2) \tag{18}$$

where

$$\beta(g^2) = -ag^4 - ba^2g^6 - ca^3g^8 + \ldots \tag{19}$$

is the G-L function,

$$\psi(ag^2) \equiv \int_0^{g^2} dg^2 \left\{ \frac{1}{\beta(g^2)} + \frac{1}{ag^4} - \frac{b}{g^2} \right\} . \tag{20}$$

(We used relation (1) between μ and g_o).

The function $\psi(ag^2)$ is analytic at $g^2=0$

$$\psi(ag^2) = (c - b^2)ag^2 + O(g^4) \tag{21}$$

so that one may put $g_o = 0$ in the last line of (18).

$$\xi = -\frac{1}{ag_R^2} - b\ \ell n\ ag_R^2 - c - b^2)ag_R^2 + O(g_R^4) . \tag{22}$$

Now, the extremum condition (15) reads

$$-\beta(g_R^2)\frac{df_i}{dg_R^2} = \alpha^2 \tag{23}$$

and we find by iterations

$$\begin{aligned} ag_R^2 = {} & \frac{\alpha}{\sqrt{b_i}} - \frac{\alpha^2}{b_i}\left(\frac{c_i}{b_i} + \frac{b}{2}\right) + \\ & + \frac{\alpha^3}{2b_i\sqrt{b_i}}\left\{\frac{b^2}{2} + \frac{4bc_i}{b_i} - \frac{1}{4}\left(\frac{c_i}{b_i} + \frac{b}{2}\right)^2 + \right. \\ & \left. + \frac{6c_i^2 - 3d_ib_i + (b^2 - c)b_i^2}{b_i^2}\right\} + O(\alpha^4) \quad . \end{aligned} \tag{24}$$

Substitution of (24) into (22), (17) and (13) yields

$$\begin{aligned} \phi_i(\alpha) = {} & b\alpha^2 \ln\alpha + a_i + 2\sqrt{b_i}\,\alpha + \\ & + \left(\frac{c_i}{b_i} - \frac{b}{2}\ln b_i\right)\alpha^2 + \\ & + \frac{\alpha^3}{\sqrt{b_i}}\left(\frac{d_i}{b_i} + c - b^2 - \left(\frac{c_i}{b_i} + \frac{b}{2}\right)^2\right) + O(\alpha^4) \, . \end{aligned} \tag{25}$$

There is only one singular term

$$b\alpha^2 \ln\alpha \tag{26}$$

which arises from $\ln a g_R^2$ in (22). Since tnis term is absent at α=1, we may drop it, and we arrive at the α-expansion.

It is important, that the α-expansion is universal, it does not depend on the choice of renormalization procedure. If we change the renormalization procedure, the coupling constant will change as follows:

$$ag_R^2 \to ag_R^2 + z_1(ag_R^2)^2 + z_2(ag_R^2)^3 + \ldots \tag{27}$$

which leads to the transformation of the terms in ordinary perturbation theory

$$a \to a \tag{28}$$

$$b \to b \tag{29}$$

$$c \to c + z_1^2 + bz_1 - z_2 \tag{30}$$

$$a_i \to a_i \tag{31}$$

$$b_i \to b_i \tag{32}$$

$$c_i \to c_i + z_1 b_i \tag{33}$$

$$c_i \to d_i + 2z_1 c_i + z_2 b_i \quad . \tag{34}$$

One may verify that under these transformations in (25)

$$\phi_i(\alpha) \to \phi_i(\alpha) + z_1 \alpha^2 \quad . \tag{35}$$

This universal change of ϕ_i can be compensated by redefinition of normalization point μ in (16).

$$\mu \to \mu \exp(-z_1 \alpha^2) \quad . \tag{36}$$

This invariance holds in all orders in α and reflects the fact that the minimal value of the function does not depend on the choice of variable.

Let us now discuss the general properties of $\phi_i(\alpha)$. At small α this function increases, since $\sqrt{b_i}$ is positive. Certainly, b_i itself should be positive for all quantum numbers i. This condition is discussed below.

From (14), (15) we see, that there is a maximum of $\phi_i(\alpha)$ at

$$\alpha = \sqrt{f_i(0)} \tag{37}$$

provided this quantity is less than 1.

As α tends to 1, $\xi(\alpha)$ increases, so that $\phi'(\alpha)$ tends to $-\infty$. The structure of singularities at $\alpha=1$ depends on the rate of the function $f_i(\xi) - \xi$ at $\xi \to +\infty$. It is natural to assume, that this function approaches its limit as some power of μR, or faster, in which case $\phi_i(\alpha)$ possesses a universal and rather weak singularity, which cancels in the mass ratios. The resulting behavior of $\phi_i(\alpha)$ is shown in Fig. 3.

The vital question concerns the convergence of the α-expansion. Does it converge around $\alpha=0$, or is it an asymptotic expansion? The possibility of convergence is supported by the recent study of Neveu, Nussinov, Koplik,[5] who proved that the number of planar diagrams increases with the order only by an exponential law (whereas the total number of diagrams increases as a factorial).

It might appear that the contributions of individual planar diagrams of the high order increase by themselves as a factoral. Here the domain of the small momenta $q_i << k$ in the loops looks dangerous. If the relevant momenta decrease with order n by a power law, then indeed the contribution of the diagram will increase as factorial

$$\left(\frac{k}{q_{\text{average}}}\right)^n \sim n^n \quad . \tag{38}$$

However, if we exclude the dangerous domain, say, by introducing a bag of size R then the planar diagrams would be bounded and the perturbation expansion is likely to be convergent. (The upper limit Λ is irrelevant due to renormalizability.) As we have seen above, the nonanalytic dependence of the masses on the bare coupling constant appears automatically at $R \to \infty$. The divergence of the perturbation series at finite R is not required for that. If the perturbation series is convergent, then the function f_i and β would be analytic at $g^2=0$, which implies analyticity of $\phi_i(\alpha) - b\alpha^2 \ell n\alpha$.

Strictly speaking Pade regularization is not equivalent to a bag. If it appears that this regularization does not provide convergence of the perturbation series, then apparently one should consider some other method. Note, however, that the general framework of an α-expansion does not rely upon the specific form of the infrared regularization, provided it is Lorentz-invariant and provided the masses decrease with R.

III. THE ZEROTH AND THE FIRST TERMS OF α-EXPANSION

In this and in the next section we find explicit expressions for the coefficients of the α-expansion in terms of Feynman diagrams of QCD. The effects oı chiral symmetry breaking will be neglected, so that our expressions would make sense only for sufficiently heavy mesons.

The zeroth and the first terms of the α-expansion are, in fact, already known. According to Ref. 4

$$a_i = \ell n\, z_s(\nu_i) \tag{39}$$

$$b_i = \gamma_i' \frac{z_s'(\nu_i)}{z_s(\nu_i)} \tag{40}$$

$$\nu_i = \Delta_i - 2 \tag{41}$$

$z_s(\nu)$ being the positive root of the Bessel function $J_\nu(2z)$, Δ_i the lowest normal dimension of the interpolating field for the given meson i, $\gamma_i' a g^2$ its anomalous dimension in the first order in coupling constant.

As far as the interpolating fields with higher dimensions are concerned, according to Ref. 4 those mix with the basic field only in the order g^8, i.e., α^4.

For example, for the Regge recurrences of the vector meson, the basic field has the form

$$V_n = \frac{1}{\sqrt{N_C}} \bar{\Psi} \gamma_\mu \xi^\mu (\xi \nabla)^{n-1} \Psi \tag{42}$$

where ξ is a lightlike vector,

$$\nabla_\mu = \partial_\mu + B_\mu \tag{43}$$

is the covariant derivative, acting on the quark field ψ. A sum over colors at fixed number of flavors of quarks is understood. In this case

$$\Delta = n + 2 \tag{44}$$

$$\gamma' = \frac{3}{11}\left(1 - \frac{2}{n(n+1)} + 4 \sum_{j=2}^{n} \frac{1}{j}\right). \tag{45}$$

As is well known, the Bessel function has an infinite number of roots $\pm z_s(\nu)$, ordered by a magnitude. All of them increase with index ν, so that the positivity of b_i is provided by the positivity of the anomalous dimensions γ_i'. For the above operators anomalous dimensions are positive for for $n>1$. The case $n=1$ (conserved vector current) is exceptional. But in this case the neglected effects of chiral symmetry breaking should be most important, since the masses of vector mesons are sufficiently low.

The vacuum trajectory corresponds to the basic fields

$$\theta_n = \frac{1}{N_C} T_r\left(\xi^\mu F_\mu^\alpha (\xi\nabla)^{n-2} F_\alpha^\nu \xi_\nu\right) \tag{46}$$

where

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + \left[B_\mu B_\nu\right] \tag{47}$$

in the usual field strength;

$$\nabla_\alpha F = \partial_\alpha F + \left[B_\alpha F\right] \tag{48}$$

is the corresponding covariant derivative. In this case

$$\Delta = n + 2 \tag{49}$$

$$\gamma' = \frac{6}{11}\left[\frac{1}{3} - \frac{4}{n(n-1)} - \frac{4}{(n+1)(n+2)} + 4\sum_{j=2}^{n} \frac{1}{j}\right]. \tag{50}$$

Here the anomalous dimensions are positive for $n > 2$. The case $n=2$ of stress-energy tensor is also an exception, until the effects of chiral symmetry breaking are taken into account.

Finally for the scalar trajectory

$$S_n = \frac{1}{\sqrt{N_C}} \bar{\psi} (\xi\nabla)^n \psi \tag{51}$$

$$\Delta = n + 3 \tag{52}$$

$$\gamma' = \frac{3}{11}\left[1 - \frac{4}{n+1} + 2 \sum_{j=2}^{n+1} \frac{1}{j}\right]. \tag{53}$$

In this case the scalar (n=0) and vector (n=1) mesons are exceptional.

The trajectories with anomalous parity, which correspond to γ_5 matrix between $\bar{\psi}$ and ψ or dual field strength $\tilde{F}_{\mu\nu}$, behave on a similar fashion. The anomalous dimensions always appear to be positive for sufficiently large n due to the term

$$\sum_{j=2}^{n} \frac{1}{j} \to \ell n\, n \tag{54}$$

which arise because of the term B_μ in covariant derivatives.

Apparently, the theory meets our expectations. A priori it was not clear, that the coefficients b_i would appear to be positive even for heavy mesons.

IV. α^2 AND α^3 TERMS

Let us now find two more terms of α-expansion. For that we need the coefficients c_i and d_i, i.e., the terms $\sim g^4$ and g^6 in the masses. In principle, we may use the general equations of Ref. 4, but for readers convenience we rederive here these equations in a simpler form. Since we are not interested in terms of g^8 and higher, we may construct the ordinary, rather then the matrix Pade approximant to the 2 point functions. (As it was shown in Ref. 4, for the conformal tensors $O_i(x)$ the nondiagonal matrix elements display themselves only starting from g^8.)

Let us consider the 2-point function of the conformal tensor $O^i_{\{\alpha\}}(x)$ in the momentum space and let us project out the states with definite spin L in the center of mass frame $k_\mu = (\sqrt{t}, 0, 0, 0)$.

The corresponding partial amplitude will be denoted as d(t) and its spectral function as $\rho(t)$, skipping the labels L, i. We shall expand d(t) in Taylor series near $t = -\Lambda^2$ where $\Lambda^2 >> \mu^2$, and then we convert the Taylor series into the diagonal Pade approximant

$$\frac{P_N(t)}{Q_N(t)} = d(t) + O(t+\Lambda^2)^{2N+1} . \tag{55}$$

The Pade equations for polynomials P_N, Q_N

$$\left(\frac{d}{dt}\right)^\ell (Q_N d - P_N)_{t=-\Lambda^2} = O, \quad \ell = 0, \ldots 2N \tag{56}$$

can be written by means of dispersion relations as follows

$$P_N(t) = Q_N(t)d(t) - \int_0^\infty \frac{ds Q_N(s)\rho(s)}{\pi(s-t)} \left(\frac{t+\Lambda^2}{s+\Lambda^2}\right)^{2N+1} \tag{57}$$

$$\int_0^\infty \frac{dt Q_N(t)\rho(t)}{(t+\Lambda^2)^{2N+1-r}} = 0, \quad r = 0, \ldots N-1 . \tag{58}$$

The problem is reduced to the solution of integral equation (58) for the denominator Q_N. Its roots determine the spectrum at given N. As we shall see below, the quantity

$$R = N/\Lambda \tag{59}$$

would play the role of the infrared cutoff. We are going to find a perturbation expansion for Q_N

$$Q_N = Q_N^0 + Q_N^{(1)} ag_R^2 + Q_N^{(2)}(ag_R^2)^2 + \dots \tag{60}$$

given the perturbation expansion for ρ

$$\rho = \rho^0 + \rho^{(1)} ag_R^2 + \rho^{(2)}(ag_R^2)^2 + \dots \tag{61}$$

By means of the Callan-Symanzik equation

$$\left(-R\frac{\partial}{\partial R} + \beta\frac{\partial}{\partial g_R^2} + 2\gamma\right)\rho = 0 \tag{62}$$

we may represent ρ as follows

$$\rho = t^{\Delta-2}(tR^2)^{\gamma + \frac{\beta}{2}\frac{\partial}{\partial g_R^2}}\,\sigma(ag_R^2) \tag{63}$$

Here Δ is the normal dimension of the field $O_i(x)$, $\gamma(ag_R^2)$ is the anomalous dimension. The function $\sigma(ag_R^2)$ coincides with the spectral density ρ at $t = R^{-2}$ and starts from the constant $\sigma(0)$ at $g_R^2 \to 0$. (There are special cases of conserved conformal tensors with $\Delta = n+2$, where for $L = n\sigma(0) = 0$, but we do not consider those for the time being.)

It would be convenient to use the "interaction representation"

$$\exp(x(\gamma + \frac{1}{2}\beta\partial)) = \exp(\gamma x)\ t\ \exp(\frac{\beta}{2}\int_0^x dy\ (a\gamma' y + \partial)) =$$

$$= \exp(\gamma x)\ 1 + \frac{a\beta}{2}\gamma' \frac{x^2}{2} + \frac{\beta}{2} x\partial + O(\beta^2) \quad . \tag{64}$$

Since $\beta \sim g_R^4$, up to g_R^8 we may write instead of (64), (63)

$$\rho = \text{const}\ t^{\nu}(1 + \frac{a\beta\gamma'}{4}\ \ell n^2(tR^2) + O(g_R^8)) \tag{65}$$

$$\nu = \Delta - 2 + \gamma + a\beta\sigma'/2\sigma \quad . \tag{66}$$

The common constant factor in (65) drops out on the homogeneous equation (58).

In order to find the perturbation theory for Q it is convenient to use the Green's function,[4] which satisfies the equation (in our notations)

$$\int_0^\infty \frac{ds\ s^{\nu} G(s,t)}{(s+\Lambda^2)^{2N+1-r}} = \frac{t^{\nu}}{(t+\Lambda^2)^{2N+1-r}}$$

$$r = 0, \dots N-1 \quad . \tag{67}$$

The explicit expression for G reads

$$G(s,t) = \oint_C \oint_{C'} \frac{d\omega\ d\omega'}{(2\pi i)^2} \frac{1}{(\omega'-\omega)} \cdot \frac{f(\omega)}{f(\omega')} \frac{(1+s/\Lambda^2)^{\omega}}{(1+t/\Lambda^2)^{\omega'}} \tag{68}$$

with

$$f(\omega) = \frac{\Gamma(2N+1-\omega)\Gamma(-\omega)}{\Gamma(N+1-\nu-\omega)\Gamma(N+1-\omega)} \cdot (N^2)^{1-\nu} \quad . \tag{69}$$

The normalization of $f(\omega)$ is the matter of convenience. The contour C encloses the poles of $f(\omega)$ which are located at $\omega = 0, \dots N$, while the contour C' encloses the zeros of $f(\omega')$ which are located at

$\omega' = N + 1 - \nu,\ N + 2 - \nu,\ \ldots,\ \infty$.

Note, that $G(s,t)$ is the Nth degree polynomial in s with t-dependent coefficients.

If we replace the factor

$$\frac{t^{\nu}}{(t+\Lambda^2)^{2N+1-r}}$$

which arises in (58), by the l.h.s. of (67), we arrive to the equation

$$\int_0^{\infty} \frac{ds\ s^{\nu}(s+\Lambda^2)^r}{(s+\Lambda^2)^{2N+1}}\ R_N(s) = 0$$

$$r = 0, \ldots N - 1 \tag{70}$$

with

$$R_N(s) = \int_0^{\infty} \frac{dt}{\Lambda^2}\ Q_N(t)\ (1 + \frac{a\beta\gamma'}{4}\ \ell n^2(tR^2)\ )G(s,t) \tag{71}$$

by construction being the Nth degree polynomial in s. As it follows from (70), this polynomial is orthogonal to all powers of s up to s^{N-1}, with the weight

$$\mathscr{W}(s) = s^{\nu}(s + \Lambda^2)^{-(2N+1)} \quad .$$

In other words R_N is the Nth degree orthogonal polynomial with respect to this weight. Up to arbitrary normalization the orthogonal polynomial is given by

$$R_N(s) = \oint_C \frac{d\omega}{2\pi i}\ f(\omega)\ (1 + s/\Lambda^2)^{\omega} \quad . \tag{72}$$

One may verify, that after integration over s in (70), (72), the arising function of ω does not have singularities outside the contour C, so that (72) is indeed a solution of (70).

Now let us expand Q in tR^2

$$Q_N = \sum_{n=0}^{N} q_n(N)\,(tR^2)^n \tag{73}$$

then substitute this expression in (71) and compare the coefficients at $(sR^2)^m$ in (71) and (72).

In (71) we find

$$\sum_{n=0}^{N} q_n(N) F_{nm}(N) \tag{74}$$

with

$$F_{nm}(N) = (N^2)^{n-m} \oint_C \oint_{C'} \frac{d\omega\, d\omega'}{(2\pi i)^2} \frac{1}{\omega' - \omega} \frac{f(\omega)}{f(\omega')} \cdot$$

$$\cdot \binom{\omega}{m} \left(1 + \frac{a\beta\gamma'}{4} \frac{\partial^2}{\partial n^2}\right) B(n+1, \omega' - n) \;, \tag{75}$$

$B(x, y)$ being the B-function. This should be equal to

$$\oint_C \frac{d\omega}{2\pi i} f(\omega) \binom{\omega}{m} (N^2)^{-m} \tag{76}$$

according to (72).

Now we may tend N to ∞ (so far, at fixed R). In this limit the region of

$$\omega \sim \omega' \sim N^2 \tag{77}$$

proves to be essential in the integrals and we may use the asymptotic form

$$f(\omega) \to \left(\frac{\omega}{N^2}\right)^{\nu-1} \exp(-N^2/\omega) \tag{78}$$

$$\binom{\omega}{m} \to \frac{\omega^m}{m!} \tag{79}$$

$$B(n+1,\omega' -n) \to \Gamma(n+1)(\omega')^{-n-1} \quad . \tag{80}$$

All the integrals can now be calculated and we find

$$\frac{(-1)^m}{m!\,\Gamma(\nu+1+m)} \tag{81}$$

in (76) and

$$F_{nm}(\infty) = \frac{(-1)^m}{m!\,\Gamma(\nu+m)} \left(1 + \frac{a\beta\gamma'}{4} \frac{\partial^2}{\partial n^2}\right) \frac{\Gamma(n+\nu)}{(m-n)\Gamma(-n)} \tag{82}$$

in (75). In the last relation the term 1 in the brackets contributes only at n=m, otherwise $\Gamma(-n)(m-n)$ is infinite. Hence we may write

$$F_{nm}(\infty) = \delta_{nm} + \frac{a\beta\gamma'}{4} \cdot \frac{(-1)^m}{m!\,\Gamma(\nu+m)} \frac{\partial^2}{\partial n^2} \frac{\Gamma(n+\nu)}{(m-n)\Gamma(-n)} \tag{83}$$

and we arrive to the following equation for coefficients $q_n(\infty) \equiv q_n$

$$q_m + \frac{a\beta\gamma'}{4} \frac{(-1)^m}{m!\Gamma(\nu+m)} \sum_{n=0}^{\infty} q_n \frac{\partial^2}{\partial n^2} \frac{\Gamma(n+\nu)}{(m-n)\Gamma(-n)} =$$

$$= \frac{(-1)^m}{m!\Gamma(\nu+m+1)} + O(g_R^8) \quad . \tag{84}$$

Up to g_R^8 we find

$$q_m = \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}\left[1 - \frac{a\beta\gamma'}{4}\,\Sigma_m(\nu)\right] \tag{85}$$

with

$$\Sigma_m(\nu) \equiv \frac{\partial^2}{\partial\epsilon^2}\bigg|_{\epsilon=0} \sum_{n=0}^{\infty} \frac{(-1)^n (m+\nu)\,\Gamma(n+\nu+\epsilon)}{n!\,\Gamma(\nu+n+1)(m-n-\epsilon)\Gamma(-n-\epsilon)} \;. \tag{86}$$

This sum can be represented as the contour integral

$$\Sigma_m(\nu) = (m+\nu)\,\frac{\partial^2}{\partial\epsilon^2}\bigg|_{\epsilon=0} \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i}\;\frac{\Gamma(-z)\Gamma(z+\nu+\epsilon)}{\Gamma(z+\nu+1)\Gamma(-z-\epsilon)m-z-\epsilon)} . \tag{87}$$

Calculating the residues in the left semiplane we find after some elementary transformations

$$\Sigma_m(\nu) = A(\nu) + 2\sum_{k=1}^{\infty}\left[\psi(k) + \psi(k+\nu)\right]\left(\frac{1}{k} - \frac{1}{k+m+\nu}\right) \;. \tag{88}$$

Here $\psi(z) = \Gamma'(z)/\Gamma(z)$, and

$$A(\nu) = \frac{\partial^2}{\partial\epsilon^2}\bigg|_{\epsilon=0} \frac{\Gamma(\nu+\epsilon)}{\Gamma(\nu)\Gamma(1-\epsilon)} \tag{89}$$

is irrelevant constant, which leads to renormalization of q_m in (85).

The corresponding equation for the mass spectrum

$$\sum_{m=0}^{\infty} \frac{(-M^2R^2)^m}{m!\,\Gamma(m+\nu+1)} \cdot \left(1 - \frac{a\beta\gamma'}{4}\,\Sigma_m(\nu)\right) = 0 \tag{90}$$

is solved iteratively and up to g^8 we find

$$MR = z_s(\nu)\Big(1 + a\beta\gamma' V_s(\nu)\Big) \tag{91}$$

with

$$V_s(\nu) = \sum_{m=0}^{\infty} C_{ms}(\nu)\Sigma_m(\nu)/8 \sum_{m=1}^{\infty} mC_{ms}(\nu) \tag{92}$$

$$C_{ms}(\nu) = \frac{(-z_s^2(\nu))^m}{m!\Gamma(m+\nu+1)} \tag{93}$$

$z_s(\nu)$ being the root of Bessel function $J_\nu(2z)$. Apparently

$$\sum_{m=0}^{\infty} C_{ms}(\nu) = 0 \tag{94}$$

so that the constant terms in $\Sigma_m(\nu)$ do not contribute. For integer ν, $\Sigma_m(\nu)$ can be reduced to the following

$$\Sigma_m(\nu) = \text{const} + 2 \sum_{\ell=1}^{\nu} \frac{\psi(\ell)}{\ell+m} +$$

$$+ \psi^2(m+\nu+1) + \psi^2(m+1) - \psi'(m+\nu+1) - \psi'(m+1) \ . \tag{95}$$

The proof of this relation is left to the reader as an entertainment for a rainy evening. Note that both ψ and ψ' are elementary for integer argument:

$$\psi = (\ell+1) = -C_E + \sum_{k=1}^{\ell} 1/k \tag{96}$$

$$\psi' = (\ell+1) = \frac{\pi^2}{6} - \sum_{k=1}^{\ell} 1/k^2 \ . \tag{97}$$

Suppose, that we know the 3-loop coefficient C in β-function and the 2- and 3-loops terms in the anomalous dimensions

$$\gamma(ag^2) = \gamma' ag^2 + \frac{1}{2}\gamma''(ag^2)^2 + \frac{1}{6}\gamma'''(ag^2)^3 + O(g^8) \tag{98}$$

and in the spectral 2-point functions

$$\sigma(ag^2) = \sigma + \sigma' ag^2 + \frac{1}{2}\sigma''(ag^2)^2 + O(g^6) \quad . \tag{99}$$

Then we may find the coefficients c_i and d_i. After simple transformations we arrive to the following expressions:

$$\frac{c_i}{b_i} = \frac{\gamma_i''}{2\gamma_i'} - \frac{\sigma_i'}{2\sigma_i\gamma_i'} + \frac{1}{2}\gamma_i' \frac{(\ln z)''}{(\ln z)'} - \frac{V}{(\ln z)'} \tag{100}$$

$$\frac{d_i}{b_i} = \frac{1}{2\gamma_i'}\left(\frac{\gamma_i'''}{3} - \frac{b\sigma_i'}{\sigma_i} - \frac{\sigma_i''}{\sigma_i} + \left(\frac{\sigma_i'}{\sigma_i}\right)^2\right) + \frac{(\ell n\, z)''}{2(\ell n\, z)'}\left(\gamma_i'' - \frac{\sigma_i'}{\sigma_i}\right) + \frac{1}{6}\gamma_i'^2 \frac{(\ell n\, z)'''}{(\ell n\, z)'} - \frac{V}{(\ell n\, z)'}\left(b + \frac{\gamma_i''}{\gamma_i'}\right) - \frac{V'}{(\ell n\, z)'}\gamma_i' \tag{101}$$

with $z \equiv z_s(\Delta-2)$, $z' \equiv z_s'(\Delta-2)$, $V \equiv V_s(\Delta-2)$, etc. Thus, the α^2 and α^3 terms are given by (25), (19), (6), (100) and (101).

The Bessel coefficients

$$L_s(\Delta) \equiv \ell n\, z \tag{102}$$

$$M_s(\Delta) \equiv 2\sqrt{(\ell n\, z)'} \tag{103}$$

$$N_s(\Delta) \equiv \frac{1}{2}\frac{(\ell n\, z)''}{(\ell n\, z)'} \tag{104}$$

$$O_s(\Delta) \equiv V/(\ell n\, z)' \tag{105}$$

are given in the Table, for the interval of s, Δ, such that

$$1.9 < z < 6 \quad . \tag{106}$$

Thus, the problem is reduced to calculation of 2- and 3-loop integrals of the ordinary perturbation theory. Though these integrals are pure numbers in our case, the calculation is rather tedious and will not be done here. However, the α^2 terms for the ratios of the radial excitations (s = 2, 3, ...) to the basic mass do not involve the 2-loop integrals. In this case

$$\ell n \frac{M_{is}}{M_{i1}} = \delta L_s + \alpha\sqrt{\gamma_i'}\,\delta M_s + + \alpha^2\left(\frac{-51}{242}\,\ell n\,\frac{M_s(\Delta)}{M_1(\Delta)} + \gamma_i'\delta N_s - \delta O_s\right) + O(\alpha^3) \tag{107}$$

where $\delta L_s \equiv L_s(\Delta) - L_1(\Delta)$, etc.

For the family of vector meson the numbers are as follows:

$$\frac{M_2}{M_1} = 1.64 - 0.26\,\alpha - 0.49\,\alpha^2 + O(\alpha^3) \tag{108}$$

for Δ=4 (spin 2) ,

$$\frac{M_2}{M_1} = 1.53 - 0.23\,\alpha - 0.37\,\alpha^2 + O(\alpha^3) \tag{109}$$

for Δ=5 (spin 3) ,

$$\frac{M_2}{M_1} = 1.46 - 0.19\,\alpha - 0.31\,\alpha^2 + O(\alpha^3) \tag{110}$$

for Δ=6 (spin 4) .

As one may see, the coefficients are rather small, but so far, do not decrease so that the α^3 terms might be important.

V. SUMMARY AND DISCUSSION

Let us resume the results obtained. We represented the ratios of masses of mesons in multicolor QCD as the boundary values at $\alpha=1$ of certain functions with the calculable Taylor coefficients. The first three coefficients were calculated explicitly. Apparently two more coefficients are required. These coefficients involve the Feynman integrals with 3 and 4 loops, which may be calculated by means of computers.

Also, the effects of the chiral symmetry breaking should be taken into account. The preliminary analysis by Yu. Makeenko and the author indicates that these effects can be fitted within the framework of α-expansion, though the coefficients will change. Thus, the above formulas for the coefficients are subject to corrections, coming from the chiral symmetry breaking. Presumably these corrections will be small for heavy mesons, and will improve the α-expansion for the light mesons. One may estimate the corrections as

$$\frac{\langle\bar{\psi}\psi\rangle^2}{M^6}$$

where $\langle\bar{\psi}\psi\rangle$ is the vacuum average, violating the chiral symmetry, and M is the mass of the corresponding meson.

The other important problem is how to compute the $1/N_c$ corrections. The method is clear in principle -- one should iterate the unitarity conditions in $1/N_c$, starting from the "Born term" found above. However, to do that one should first find the Pade-regularization for the many-point functions, which meets the technical difficulties.

After this problem would be solved, one should think about the exponential corrections $\sim \exp(-N_c)$ due to the instantons and the other possible extrema of the classical action. The related problem is the problem of baryons, which are composed from N_c quarks and thus have the infinite masses in the limit $N_c = \infty$.

Thus, the way proposed is not an easy one, but there seems to be a light at the end of the tunnel.

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TABLE

Δ	s	L	M	N	O
	1	1.343310	1.186201	-0.217058	0.992370
3	2	1.948134	0.906111	-0.133893	1.487123
	3	2.319783	0.762349	-0.096919	1.806861
	1	1.636201	0.994074	-0.145955	1.367648
4	2	2.130282	0.806521	-0.101979	1.752972
	3	2.452714	0.698700	-0.078759	2.016917
	1	1.853193	0.875952	-0.110426	1.630258
5	2	2.278397	0.735916	-0.082651	1.953043
	1	2.026613	0.793436	-0.089024	1.833420
6	2	2.403761	0.682224	-0.069636	2.114700
7	1	2.171506	0.731392	-0.074680	1.999564
8	1	2.296175	0.682444	-0.064379	2.140340
9	1	2.405716	0.642491	-0.056614	2.262600

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FIGURE CAPTIONS

Fig. 1 Logarithm of the generalized mass versus $\xi = \ell n \mu R$, R being the I. R. cutoff. When α varies from 0 to 1, the curve rotates around $\xi = 0$.

Fig. 2 The relation between the position ξ_o of the minimum of the mass and α. Varying α from 0 to 1 is equivalent to varying ξ_o from $-\infty$ to $+\infty$.

Fig. 3 The relation between the minimal value ϕ of the log of mass and α. $\phi(\alpha)$ starts from f(0) and ends up with the true value of $\ell n(M/\mu)$. The expansion of ϕ around $\alpha = 0$ is derived in a text.

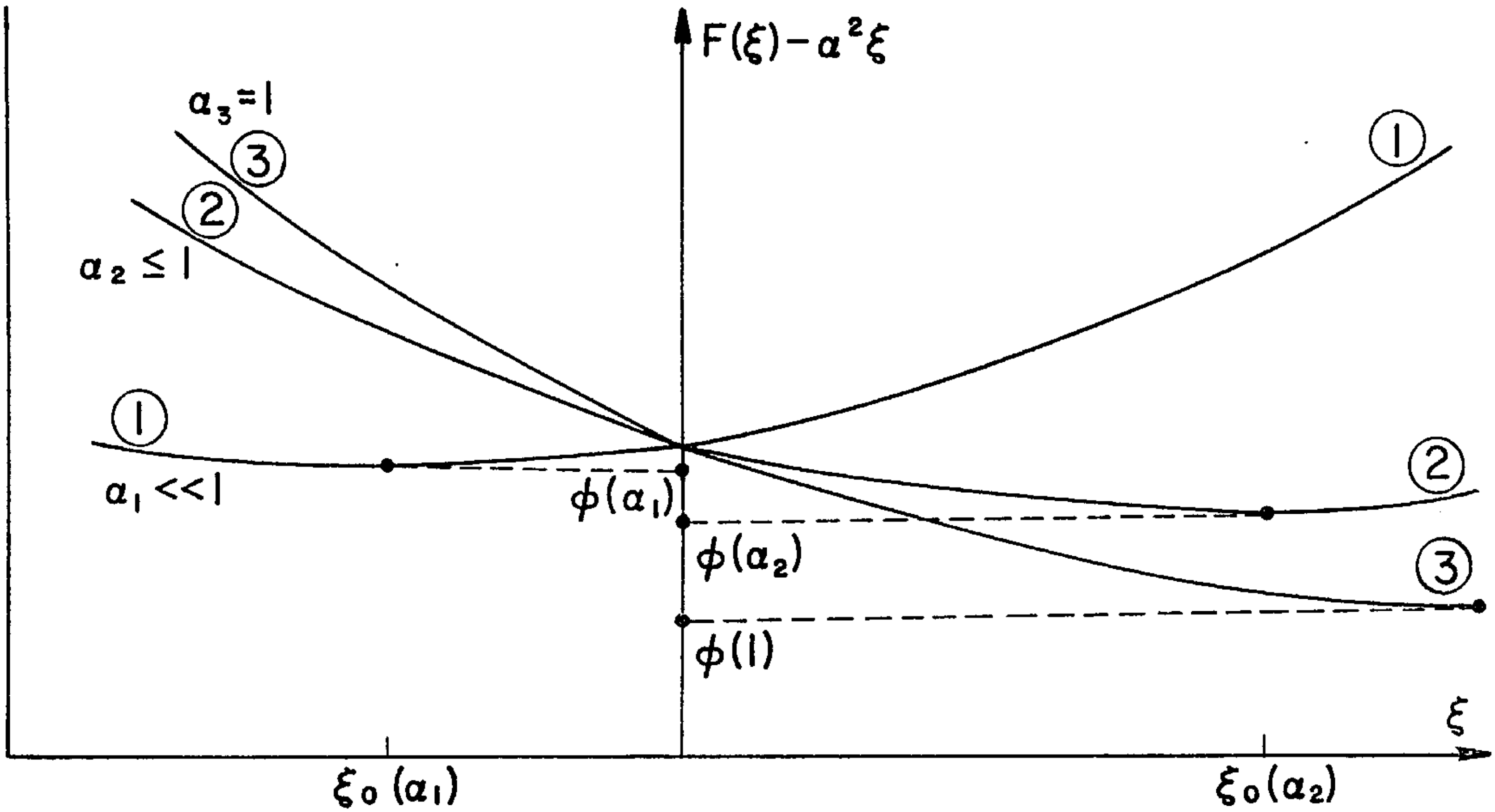

Figure 1

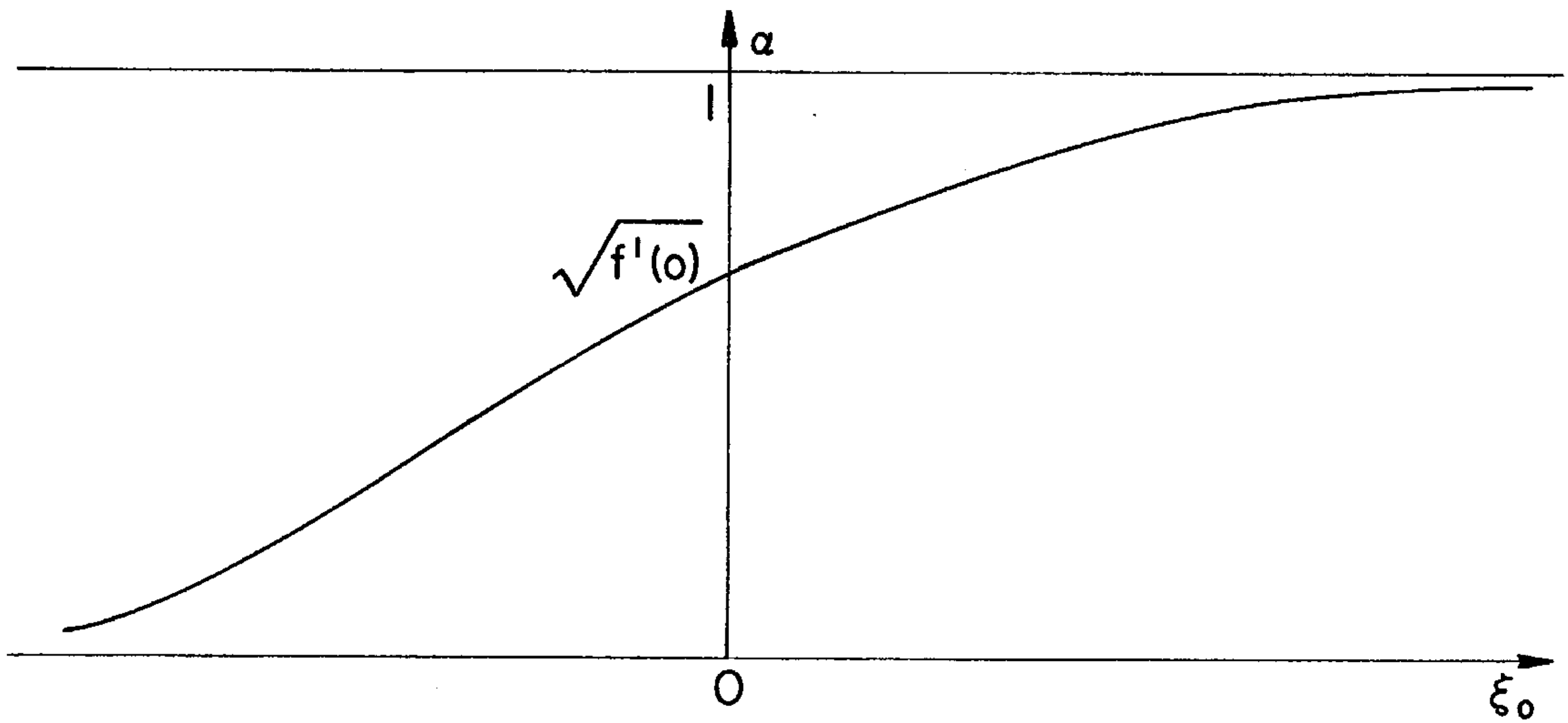

Figure 2

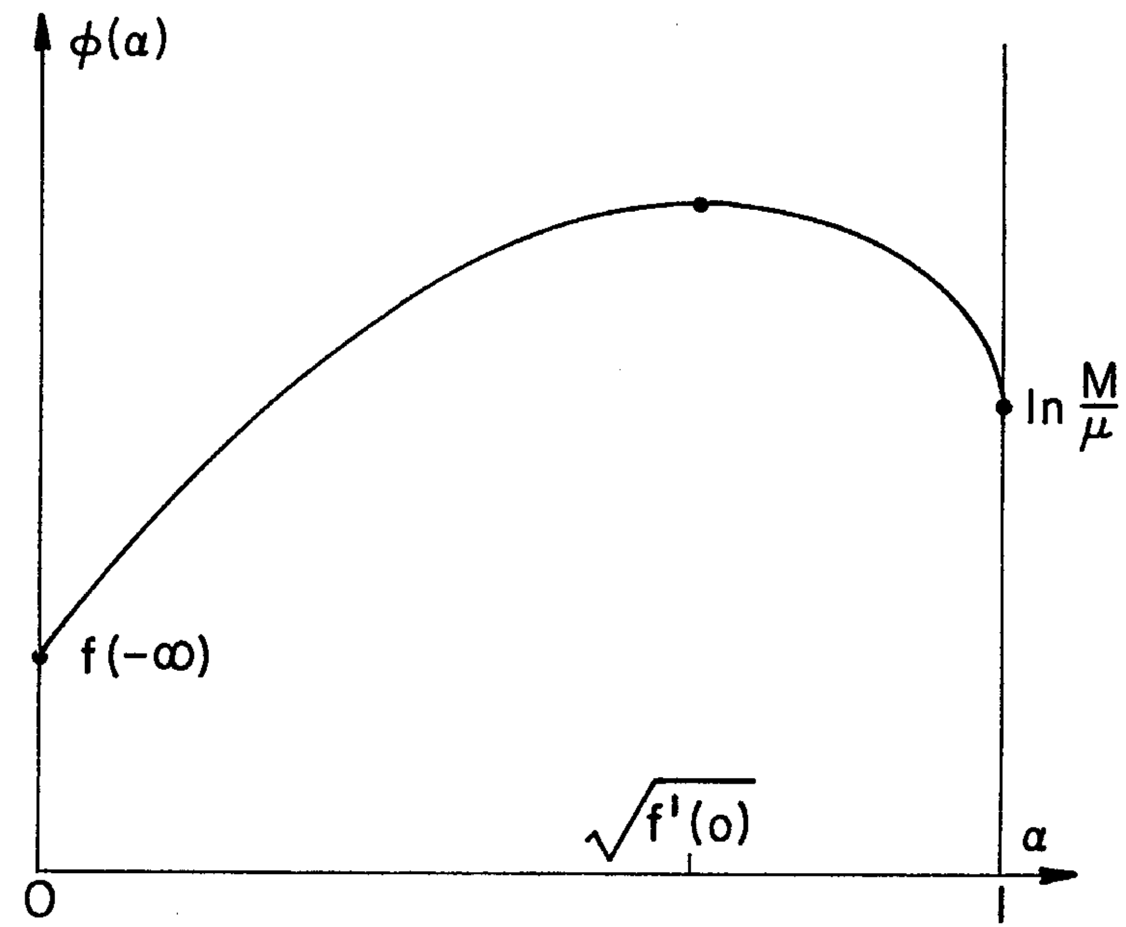

Figure 3